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Construction of new nonlinear dynamical systems on the basis of known ones

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Abstract. The ways for constructing continuous in time nonlinear dynamical systems in the form of sets of nonlinear ordinary differential equations could be classified as direct and indirect. In the first case the equations are constructed directly as a model of some particular phenomenon under consideration or on the basis of pure mathematical reasons. In the second case, they are product of some kind of expansion (e.g. Gallerkin approximation) of originally derived partial differential equations and subsequent truncation. Countless number of such systems are known at present. The most interesting phenomenon related to them is the possibility of having chaotic solutions for some values of the parameters entering into the equations. Hence, a possibility for control of their behaviour by varying the parameters exists. This problem is a subject of the present paper. An approach to it is developed and applied to the classical Lorenz and Rössler systems. Some elements of the general approach presented here, can be found in earlier publications of other authors. In this paper only analytical tools are used.

Keywords: nonlinear dynamical systems (NDSs), modified Lorenz and Rössler systems, coupled NDSs, stability properties control

1. INTRODUCTION AND GENERAL FORMULATION

A continuous in time Nonlinear Dynamical System (NDS) is usually represented by a set of first order nonlinear ordinary differential equations (ODEs)

$$\dot{X}_k = dX_k / dt = F_k(X_i(t), r_j) \tag{1}$$

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where $i,k=\overline{1,n}$, $j=\overline{1,m}$, F is a nonlinear (vector) function, r_j are parameters. Completely integrable NDSs can have regular behavior only, while for the nonitegrable ones, chaotic behaviour is possible at $n \geq 3$. Moreover, one and the same NDS can behave regularly for one set of values of its parameters r_j and chaotically for another.

A given NDS can be perturbed by additive forcing of the type

$$\dot{X}_k = F_k(X_i, r_j) + H_k(t) \tag{2}$$

where $H_k(t)$ are prescribed (most frequently periodic) functions of time *t*. Another way of perturbation consists in introducing of own signals to the system's parameters r_j by replacing them with variable ones

$$r_j \to r_j(X_i(t)) \tag{3}$$

It is consistent to call this procedure self-perturbation (excitation) of the NDS. If two different NDSs are considered

$$\dot{X}_k = F_k(X_i(t), r_j)\ ,\ \dot{Y}_k = G_k(Y_i(t), s_p) \tag{4}$$

where $p=\overline{1,m'}$ and

$$r_j \to r_j(Y_i(t))\ ,\ s_p \to s_p(X_i(t)) \tag{5}$$

is assumed, then mutual "two-way" perturbation will take place. As a particular case, the role of the second system in (4) may be played by the first one but for different set of values of r_j. Finally, mixed feedback is also possible:

$$r_j \to r_j(Y_i(t), X_i(t))\ ,\ s_p \to s_p(X_i(t), Y_i(t)) \tag{6}$$

Appropriately used, the substitutions (3), (5), (6) can introduce more nonlinearity in the original systems (1) and (4) and give rise to a class of new NDSs in each particular case. The main question to be answered is how does the global behaviour of (2) or (4) changes as a result of the above described intervention. In other words, this means control of the system's behaviour – a problem that attracted and is still attracting many researchers. Due to the wide potential applications of chaos control, various synchronisation schemes have been proposed in the last two decades both in theoretical analysis and experimental implementations (Chen, 1994), (Chen&Dong, 1993), (Nijmeijer, 1997), (Sundarapandiana&Pehlivan, 2012), (Kountchou et al, 2017).

Since chaotic behaviour of a NDS is likely at $n \geq 3$ only, we shall consider here the simplest case $n = 3$. Most of the NDSs studied so far are 3-dimensional. No doubt, two of them are classical examples of very simple deterministic systems capable of generating chaos, namely the Lorenz (1963) system

$$\begin{aligned} \dot{X} &= \sigma(Y - X) \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= -bZ + XY \end{aligned} \tag{7}$$

and the Rössler (1976) system

$$\begin{aligned} \dot{X} &= -Y - Z \\ \dot{Y} &= X + aY \\ \dot{Z} &= b_0 + bX - cZ + XZ \end{aligned} \tag{8}$$

Henceforth usual notations (*X, Y, Z*) will be used instead of (X_1, X_2, X_3) . Obviously, $r_j = (\sigma, r, b)$ for (7), while $s_p = (a, b_0, b, c)$ for (8).

Section 2 deals with the system (7), while section 3 is based on (8). In section 4, by special coupling of (7) with (8), a new 6-dimensional NDS is constructed. Finally, in section 5, an example of mixed perturbation is considered. Each section contains also discussion of the results obtained by means of analytical tools. This is the reason to pay attention only to the stationary and stability properties of the systems. Numerical studies are planned to be made in the future.

2. SELF PERTURBATION OF THE LORENZ CHAOTIC SYSTEM

2.1. Single Lorenz system (7)

Traditionally σ and b are fixed constants (10 and 8/3, respectively) and r is considered as a bifurcation parameter. The latter comes from the physical interpretation of r as Rayleigh number in the classical problem of thermal (Benard) convection in real fluids heated from below. That is why we choose to introduce a feedback into (7) by means of this parameter based on the general idea (3) $r \to r(X, Y, Z)$. The simplest clear form would be the linear part of the Taylor expansion

$$r(X, Y, Z) = r + kX(t) + mY(t) + nZ(t) \tag{9}$$

where $r = const > 0$ while (k, m, n) are new free constants by means of which one can manipulate the original system (7). Alternative possibilities are to apply the same approach to σ or b, or simultaneously to all of them.

Thus, we shall consider the modified system

$$\begin{aligned} \dot{X} &= \sigma(Y - X) \\ \dot{Y} &= (r + kX + mY + nZ)X - Y - XZ \\ \dot{Z} &= -bZ + XY \end{aligned} \tag{10}$$

The nonzero stationary solutions (fixed points in the phase space X, Y, Z) in this case are:

$$\begin{aligned} &\overline{X} = \overline{Y}, \qquad \overline{Z} = \overline{Y}^2 / b, \\ &(n-1)\overline{Y}^2 + sb\overline{Y} + b(r-1) = 0 \end{aligned} \tag{11}$$

where $s = k + m$ and the real roots of the quadratic equation for $\overline{Y}$ are meaningful only. Besides, a trivial (zero) stationary solution always exists

$$\overline{X} = \overline{Y} = \overline{Z} = 0 \tag{12}$$

The stability properties of (12) do not depend on (k, m, n), i.e. it is stable at $r < 1$ and unstable at $r > 1$. As for the stability of the nonzero solutions (11), the following cubic characteristic equation is in force

$$\begin{aligned} \lambda^3 + \left[(\sigma + b + 1) - m\overline{Y}\right]\lambda^2 + \left[b(\sigma + r) - (\sigma s - kb)\overline{Y}\right]\lambda + \\ \sigma s b\overline{Y} + 2\sigma b(r-1) = 0 \end{aligned} \tag{13}$$

The presence of additional free parameters (k, m, n) allows one to control the stationary states and their stability properties. Two examples are considered below.

2.1.1. The case with $n = 1, s \neq 0$

In this case there exists one nonzero fixed point only for all $0 < r < \infty$, namely

$$\overline{X} = \overline{Y}, \qquad \overline{Z} = \overline{Y}^2 / b, \quad \overline{Y} = (1 - r) / s \tag{14}$$

and the equation (13) degenerates into $\lambda = -b$ and

$$\lambda^2 + \left((\sigma+1) - \frac{1-r}{1+\delta} \right)\lambda - \sigma(1-r) = 0 \tag{15}$$

where $\delta = k/m$. Hence $\lambda = \lambda(\sigma,\delta,r)$. If $\sigma = 10$ and $\delta = 0$ $(k = 0$ but $m = s \neq 0)$, then

$$\lambda_\pm = -(5+\frac{r}{2}) \pm \sqrt{(5-\frac{r}{2})^2 + 10} \tag{16}$$

so that $\lambda_\pm < 0$ at $r > 1$ (*stable stationary state*) while at $r < 1$ one has $\lambda_+ > 0, \lambda_- < 0$. Therefore, the bifurcation diagram and the phase portrait of the original system (7) is globally altered at $n = 1$.

Chaotic behaviour (strange attractor) is impossible in this case, simply because of decoupling of (10) in the two-dimensional system

$$\dot{X} = \sigma(Y - X)$$
$$\dot{Y} = rX + kX^2 + mXY - Y$$

and a single equation $\dot{Z} + bZ = XY$.

2.1.2. The case with $n \neq 1, s = 0$ (i.e. $k = -m$)

Now

$$\overline{X} = \overline{Y} = \pm\sqrt{b_n(r-1)}, \quad \overline{Z} = (r-1)/(1-n) \tag{17}$$

where $b_n = b/(1-n)$ and $n > 1$ or $n < 1$. In the first case $b_n < 0$ and the solutions (17) exist only at $r < 1$. Again the phase portrait of the original system (7) is globally altered. However, if $n < 1$, then the nonzero solutions (17) exist for all $r > 1$ with the stability properties governed by the equation (13). In the particular case $n = 0$ and $k = -m \neq 0$, expressions (17) are exactly the Lorenz system's solutions $\overline{X} = \overline{Y} = \pm\sqrt{b(r-1)}, \overline{Z} = r - 1$, but according to (13) the eigenvalue equation is

$$\lambda^3 + \left[(\sigma+b+1) - m\overline{Y}\right]\lambda^2 + \left[b(\sigma+r) - m\overline{Y}\right]\lambda + 2\sigma b(r-1) = 0 \tag{18}$$

It follows that each solution for $\overline{Y}$ has its own characteristic equation and consequently different stability properties.

In the general case ($n \neq 1,\ s \neq 0$), equations (11) and (13) are in force and have to be analyzed for existence of real solutions and their stability. The results are expected to be intermediate compared to the previous two cases.

2.2. Two coupled Lorenz systems

Two Lorenz systems are considered

$$\begin{aligned} \dot{X}_1 &= \sigma_1(Y_1 - X_1) & \dot{X}_2 &= \sigma_2(Y_2 - X_2) \\ \dot{Y}_1 &= r_1 X_1 - Y_1 - X_1 Z_1 & \dot{Y}_2 &= r_2 X_2 - Y_2 - X_2 Z_2 \\ \dot{Z}_1 &= -b_1 Z_1 + X_1 Y_1 & \dot{Z}_2 &= -b_2 Z_2 + X_2 Y_2 \end{aligned} \tag{19 a,b}$$

where $(\sigma_i, r_i, b_i) > 0, \quad i = 1,2$ are parameters governing the behaviour (regular or chaotic) of each of them. Referring to (5) and (9) we assume

$$\begin{aligned} r_1(X_2, Y_2, Z_2) &= r_1 + k_1 X_2 + m_1 Y_2 + n_1 Z_2 \\ r_2(X_1, Y_1, Z_1) &= r_2 + k_2 X_1 + m_2 Y_1 + n_2 Z_1 \end{aligned} \tag{20}$$

Thus, the systems (19) are coupled and form a new six-dimensional NDS possessing its own dynamics. With $\dot{X}_i = \dot{Y}_i = \dot{Z}_i = 0$ we find

$$\begin{aligned} &\overline{X}_i = \overline{Y}_i, \qquad \overline{Z}_i = \overline{Y_i^2} / b_i, \qquad i = 1,2 \\ &\overline{Y}_1^2 = b_1(r_1 - 1) + s_1 b_1 \overline{Y}_2 + (n_1 b_1 / b_2)\overline{Y}_2^2 \\ &\overline{Y}_2^2 = b_2(r_2 - 1) + s_2 b_2 \overline{Y}_1 + (n_2 b_2 / b_1)\overline{Y}_1^2 \end{aligned} \tag{21}$$

where $s_i = k_i + m_i$. Eliminating either $\overline{Y}_1$ or $\overline{Y}_2$, a polynomial equation of 4th order can be derived and solved analytically. For example, if $n_1 = 0$ and $s_1 = s_2 = 0$ (i.e. $k_i = -m_i$), then

$$\overline{Y}_1 = \pm\sqrt{b_1(r_1 - 1)} \quad \text{at } r_1 > 1 \tag{22}$$

and

$$\overline{Y}_2^2 = b_2(r_2 - 1) + n_2 b_2 (r_1 - 1) > 0 \tag{23}$$

at

$$r_2 > \begin{cases} 1+|n_2|(r_1-1) > 1 \ \text{ if } \ n_2 < 0 \\ 1-n_2(r_1-1) < 1 \ \text{ if } \ n_2 > 0 \end{cases} \tag{24}$$

Similar alternative result can be obtained at $n_2 = 0, \quad n_1 \neq 0$.

3.SELF PERTURBATION OF THE RÖSSLER CHAOTIC SYSTEM

3.1. Single Rössler system (8)

Unlike (7), the Rössler system consists of two linear and one nonlinear ODEs and is one of the simplest known NDSs with chaotic behaviour. Most often the Z - equation (8) has been used with either $b = 0$ or $b_0 = 0$. In the first case

$$\overline{X} = -a\overline{Y}, \qquad \overline{Z} = -\overline{Y}, \qquad a\overline{Y}^2 + c\overline{Y} + b_0 = 0 \tag{25}$$

so that two nonzero fixed points exist:

$$\overline{Y}_{1,2} = \frac{1}{2a}(-c \pm \sqrt{c^2 - 4ab_0}) \quad \text{at} \quad c^2 > 4ab_0. \tag{26}$$

It has been proved that at $a = b_0 = 0.2$ and $c = 5.7$ the nonstationary solution is chaotic. In the second case ($b_0 = 0$) the stationary solutions are

$$\begin{aligned} &\overline{X}_1 = \overline{Y}_1 = \overline{Z}_1 = 0 \\ &\overline{X}_2 = -a\overline{Y}_2, \qquad \overline{Z}_2 = -\overline{Y}_2, \qquad \overline{Y}_2 = b - c/a \end{aligned} \tag{27}$$

It has been proved that at $a = 0.32$ or 0.38, $b = 0.3$ and $c = 4.5$ the solution is chaotic.

In our case (8), we have the equation

$$a\overline{Y}^2 - (ab - c)\overline{Y} + b_0 = 0 \tag{28}$$

under condition $(ab - c)^2 > 4ab_0$ instead of (25) and (26).

The most appropriate for perturbation appears to be the Y-equation in (8) through the bifurcation parameter a replaced by

$$a(X,Y,Z) = a + kX(t) + mY(t) + nZ(t) \tag{29}$$

where $a = const$ and (k, m, n) are free parameters. Thus, we shall study the modified Rössler system

$$\begin{aligned} \dot{X} &= -Y - Z \\ \dot{Y} &= X + (a + kX + mY + nZ)Y \\ \dot{Z} &= b_0 + bX - cZ + XZ \end{aligned} \tag{30}$$

Hence, we find the stationary solutions

$$\overline{Z} = -\overline{Y}, \quad \overline{X} = -(a\overline{Y} + \delta\overline{Y}^2)/(1 + k\overline{Y}) \tag{31}$$

$$\delta\overline{Y}^3 + (a + ck - \delta b)\overline{Y}^2 + (c + kb_0 - ab)\overline{Y} + b_0 = 0 \tag{32}$$

where $\delta = m - n$. The cubic equation (32) can have up to three real solutions. Thus, a third fixed point can be introduced compared to (28). Two particular cases are of interest:

a) Let $b_0 = 0$. Then $\overline{X}_1 = \overline{Y}_1 = \overline{Z}_1 = 0$ and the other two fixed points are determined from the real roots of the quadratic equation following from (32).

b) Let $\delta = 0,\ b_0 \neq 0$. One of the fixed points disappears – the equation for $\overline{Y}$ is quadratic. If in addition $k = -a/c$, the equation (32) becomes linear and

$$\overline{Y} = -b_0/(c + kb_0 - ab) \tag{33}$$

Therefore, the global view of the original phase portrait can be essentially altered by the proposed modification (30) of the Y-equation.

Further, on the basis of these analytical results, linear stability analysis and numerical calculations can be done.

3.2. Two coupled Rössler systems

For simplicity we shall consider here the case with $b_0 = 0$ in (8). We shall further assume that two sets of parameters $(a_1, b_1, c_1) \neq (a_2, b_2, c_2)$ define two Rössler systems

$$\begin{aligned} \dot{X}_1 &= -Y_1 - Z_1 & \dot{X}_2 &= -Y_2 - Z_2 \\ \dot{Y}_1 &= X_1 + a_1 Y_1 & \dot{Y}_2 &= X_2 + a_2 Y_2 \\ \dot{Z}_1 &= b_1 X_1 - c_1 Z_1 + X_1 Z_1 & \dot{Z}_2 &= b_2 X_2 - c_2 Z_2 + X_2 Z_2 \end{aligned} \tag{34}$$

working in different regimes. If they are coupled in some way, they will form a 6-component NDS with its own dynamics. We choose to do this by letting

$$\begin{aligned} a_1(X_2, Y_2, Z_2) &= a_1 + k_1 X_2 + m_1 Y_2 + n_1 Z_2 \\ a_2(X_1, Y_1, Z_1) &= a_2 + k_2 X_1 + m_2 Y_1 + n_2 Z_1 \end{aligned} \tag{35}$$

where (k_i, m_i, n_i), $i = 1, 2$ are free coupling coefficients (parameters) not necessarily all different from zero. If $k_1 = m_1 = n_1 = 0$ or $k_2 = m_2 = n_2 = 0$, the perturbation is "one way". Since the general case (35) is less attractive, we shall first consider some particular cases.

Because always $\overline{Z}_i = -\overline{Y}_i$, we present hereafter only the nonzero solutions for $\overline{X}_i$ and $\overline{Y}_i$. In all cases $\overline{X}_i = \overline{Y}_i = \overline{Z}_i = 0 \quad i = 1, 2$ is a stationary solution.

a) Let $m_i = n_i = 0, \quad i = 1, 2$. The stationary solutions are

$$\begin{aligned} \overline{X}_1 &= (c_1 - a_1 b_1) - k_1 b_1 \overline{X}_2 = A_{10} - k_1 b_1 \overline{X}_2 \\ \overline{X}_2 &= \frac{A_{20} - k_2 b_2 A_{10}}{1 - k_1 k_2 b_1 b_2}, \quad (A_{20} = c_2 - a_2 b_2) \\ \overline{Y}_1 &= -\frac{\overline{X}_1}{a_1 + k_1 \overline{X}_2}, \qquad \overline{Y}_2 = -\frac{\overline{X}_2}{a_2 + k_2 \overline{X}_1} \end{aligned} \tag{36}$$

Therefore, regardless of the coupling, each system (34) has only one nonzero fixed point.

b) Let $k_i = n_i = 0, \quad i = 1, 2$. Now

$$\overline{X}_1 = -(a_1 + m_1 \overline{Y}_2)\overline{Y}_1, \qquad \overline{X}_2 = -(a_2 + m_2 \overline{Y}_1)\overline{Y}_2 \tag{37}$$

where

$$\begin{aligned} (\overline{Y}_1 - b_1)(a_1 + m_1 \overline{Y}_2) + c_1 &= 0 \\ (\overline{Y}_2 - b_2)(a_2 + m_2 \overline{Y}_1) + c_2 &= 0 \end{aligned} \tag{38}$$

which means that a quadratic equation can be derived for $\overline{Y}_1$ and $\overline{Y}_2$. Therefore, unlike the previous case, each system (34) can have two nonzero real fixed points under certain conditions. Similar results could be obtained when $k_i = m_i = 0, \quad i = 1, 2$.

c) Let now $m_1 = n_1 = k_2 = n_2 = 0$. This is a kind of "cross-perturbation". For this case

$$\overline{X}_1 = -(a_1 + k_1\overline{X}_2)\overline{Y}_1, \qquad \overline{Y}_2 = -\overline{X}_2/(a_2 + m_2\overline{Y}_1) \tag{39}$$

where ($\overline{Y}_1, \overline{X}_2$) satisfy the algebraic equations

$$\begin{aligned} &(\overline{Y}_1 - b_1)(a_1 + k_1\overline{X}_2) + c_1 = 0 \\ &b_2(a_2 + m_2\overline{Y}_1) + \overline{X}_2 - c_2 = 0 \end{aligned} \tag{40}$$

and again a quadratic equation can be derived for $\overline{Y}_1$ and $\overline{X}_2$.

d) Thus, we come to **the most general case** (35) with all free parameters $(k_i, m_i, n_i) \neq 0$. Now the stationary solutions can be derived from the nonlinear algebraic system

$$\begin{aligned} &\overline{X}_1 = -(a_1 + k_1\overline{X}_2 + \delta_1\overline{Y}_2)\overline{Y}_1 \\ &\overline{X}_2 = -(a_2 + k_2\overline{X}_1 + \delta_2\overline{Y}_1)\overline{Y}_2 \\ &0 = c_1\overline{Y}_1 + (b_1 - \overline{Y}_1)\overline{X}_1 \\ &0 = c_2\overline{Y}_2 + (b_2 - \overline{Y}_2)\overline{X}_2 \end{aligned} \tag{41}$$

where $\delta_i = m_i - n_i$. Generally, equations (41) may have up to four real nonzero solutions. Their stability is governed by the following eigenvalue equation

$$\begin{vmatrix} -\lambda & -1 & -1 & 0 & 0 & 0 \\ 1 & -\overline{X}_1/\overline{Y}_1 - \lambda & 0 & k_1\overline{Y}_1 & m_1\overline{Y}_1 & n_1\overline{Y}_1 \\ b_1 - \overline{Y}_1 & 0 & \overline{X}_1 - c_1 - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & -1 & -1 \\ k_2\overline{Y}_2 & m_2\overline{Y}_2 & n_2\overline{Y}_2 & 1 & -\overline{X}_2/\overline{Y}_2 - \lambda & 0 \\ 0 & 0 & 0 & b_2 - \overline{Y}_2 & 0 & \overline{X}_2 - c_2 - \lambda \end{vmatrix} = 0 \tag{42}$$

One has to use numerical techniques to solve (41) and (42) for particular values of the parameters.

4. MUTUAL PERTURBATION OF THE RÖSSLER AND LORENZ SYSTEMS

A 6-component system can be constructed by coupling the classical systems (34a) and (19b) through the corresponding expressions (35) and (20) for $a_1(X_2,Y_2,Z_2)$ and $r_2(X_1,Y_1,Z_1)$. Despite of their complexity, the equations describing the stationary state of the system can be solved analytically:

$$\begin{aligned}
&\overline{X}_1 = c_1 - b_1(a_1 + s_1\overline{Y}_2 + (n_1/b_1)\overline{Y}_2^2)\\
&\overline{Y}_1 = b_1 - c_1(a_1 + s_1\overline{Y}_2 + (n_1/b_1)\overline{Y}_2^2)\\
&\overline{Y}_2^2 = b_2\left[(r_2-1) + k_2\overline{X}_1 + \delta_2\overline{Y}_1\right], \quad \delta_2 = m_2 - n_2\\
&\overline{Z}_1 = -\overline{Y}_1, \quad \overline{X}_2 = \overline{Y}_2, \quad \overline{Z}_2 = \overline{Y}_2^2/b_2
\end{aligned}$$

Hence, the equation for $\overline{Y}_2$ takes the form

$$A\overline{Y}_2^4 + B\overline{Y}_2^3 + C\overline{Y}_2^2 + D\overline{Y}_2 + E = 0$$

where A,B,C,D,E are coefficients such that $A = B = 0$ at $\delta_2 = 0$, $B = D = 0$ at $s_1 = 0$, $A = B = D = 0$ at $\delta_2 = s_1 = 0$ and $A = 0$ at $k_2 = n_2 = 0$.

5. MIXED PERTURBATION OF NDSs

This most general case requires considering at least two different NDSs (X_1,Y_1,Z_1) and (X_2,Y_2,Z_2), e.g. (19ab), (34ab) or (34a) coupled with (19b) in which the bifurcation/controlling parameters "*a*" and "*r*" must be replaced by the corresponding expressions (6):

$$\begin{aligned}
&r \to r + (kX_1 + mY_1 + nZ_1) + (pX_2 + qY_2 + sZ_2)\\
&a \to a + (k'X_1 + m'Y_1 + n'Z_1) + (p'X_2 + q'Y_2 + s'Z_2)
\end{aligned} \qquad (43)$$

Obviously, several particular (simpler) cases can be extracted from (43). As an example we shall consider the coupled Rössler system (34ab) with a_1 and a_2 replaced by

$$\begin{aligned}
&a_1(Y_1,Z_2) = a_1 + m_1Y_1 + n_1Z_2\\
&a_2(Y_2,Z_1) = a_2 + m_2Y_2 + n_2Z_1
\end{aligned}$$

The stationary solutions in this case are

$$\overline{X}_1 = -(a_1 + m_1\overline{Y}_1 - n_1\overline{Y}_2)\overline{Y}_1$$
$$\overline{X}_2 = -(a_2 + m_2\overline{Y}_2 - n_2\overline{Y}_1)\overline{Y}_2$$
$$\overline{Z}_i = -\overline{Y}_i, \quad (i = 1,2)$$

where $\overline{Y}_1, \overline{Y}_2$ are the real solutions of the following nonlinear algebraic system

$$(b_1 - \overline{Y}_1\ )(a_1 + m_1\overline{Y}_1 - n_1\overline{Y}_2) = c_1$$
$$(b_2 - \overline{Y}_2\ )(a_2 + m_2\overline{Y}_2 - n_2\overline{Y}_1) = c_2$$

6. CONCLUSION

We have shown that as a result of self and mutual perturbation by coupling of individual NDSs under consideration (7) and (8) trough the relationships (10), (20), (30) and (35), new fixed points in the phase space can appear and the geometry of the corresponding phase portraits can be altered considerably. The general approach briefly described in Section 1 and "illustrated" in sections 2 to 5 can be applied to any other NDSs, including such with different dimensions, e.g. coupling 3-dimensional NDS with 2-dimensional one. In essence, this is a way of constructing new NDSs with their own dynamics on the basis of known ones. As stated in Poland (1993), inverse problem can be solved – given some NDS, e.g. constructed in a way proposed here, chemical reaction could be realized obeying the properties of this new system. The results, obtained in the present paper may be also applied in the laser physics (laser array systems governed by the same Lorenz equations (7) and (19) but with different from the original physical meaning of the phase variables and parameters (Liu&Barbosa, 1995).

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